$\boxed{Ex}$ Let $S(x,y) = \dfrac{4xy}{(x^2+1)(y^2+1)} = 4\left(\dfrac{x}{x^2+1}\right)\left(\dfrac{y}{y^2+1}\right)$

Find the absolute minimum & maximum of this function on the unit square.

$\{(x,y) : x \geq 0,\ y \geq 0,\ x^2+y^2 \leq 16\}$

- Interior critical points

$\nabla S = (0,0)$

$$S_x = \frac{4y}{y^2+1}\left(\frac{x}{x^2+1}\right)' = \frac{4y}{y^2+1}\left(\frac{(x^2+1)\cdot 1 - x(2x)}{(x^2+1)^2}\right)$$

$$= \frac{4y}{(y^2+1)}\frac{(1-x^2)}{(x^2+1)^2}$$

$$S_y = \left(\frac{4x}{x^2+1}\right)\frac{(1-y^2)}{(y^2+1)^2}.$$

$$\nabla S = \left(\frac{4y(1-x^2)}{(y^2+1)(x^2+1)^2},\ \frac{4x(1-y^2)}{(x^2+1)(y^2+1)^2}\right) = (0,0)$$

$$\frac{4y(1-x^2)}{(y^2+1)(x^2+1)^2} = 0 \Leftrightarrow 4y(1-x^2) = 0 \quad (*)$$

fraction $\frac{A}{B} = 0 \Leftrightarrow A = 0.$ with $B \neq 0.$

$$\frac{4x(1-y^2)}{(x^2+1)(y^2+1)^2} = 0 \Leftrightarrow 4x(1-y^2) = 0 \quad (**)$$

$(*) \Rightarrow 4y(1+x)(1-x) = 0$

$\Rightarrow y = 0$ OR $x = -1$ OR $x = 1$

$(**) \Rightarrow 4x = 0$

$x = 0$

$(0,0)$

$-4(1+y)(1-y) = 0$

$\Rightarrow y = 1$ or $y = -1$

$(-1,1)$, $(-1,-1)$

Not in Region

$4(1+y)(1-y) = 0$

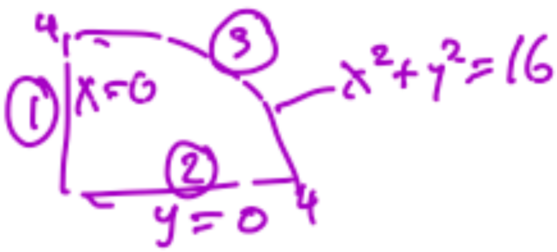
$y = 1$ or $y = -1$

$(1,1)$ $(1,-1)$

Not in region.

$(0,0)$ & $(1,1)$ are possible extrema.

Let's consider boundary pieces.



$$S(x,y) = \frac{4xy}{(x^2+1)(y^2+1)}$$

① $S_1(y) = \dfrac{4 \cdot 0 \cdot y}{(0^2+1)(y^2+1)} = 0$. $(0,y)$

$0 \le y \le 4$ are all critical

② $S_2(x) = \dfrac{4 \cdot x \cdot 0}{(x^2+1)(0^2+1)} = 0$ $(x,0)$

$0 \le x \le 4$ all critical.

③ $x^2+y^2=16$

$\uparrow$ parametrize $\quad x=4\cos\theta$

$y=4\sin\theta$

$0\le\theta\le \cancel{2\pi}\ \frac{\pi}{2}$

$$S_3(\theta)=\frac{4\cdot(4\cos\theta)(4\sin\theta)}{\left((4\cos\theta)^2+1\right)\left((4\sin\theta)^2+1\right)}$$

$$=\frac{64\cos\theta\sin\theta}{(16\cos^2\theta+1)(16\sin^2\theta+1)}$$

$$0=S_3'(\theta)=\frac{(16\cos^2\theta+1)(16\sin^2\theta+1)\left[64\cos\theta\sin\theta\right]' - 64\cos\theta\sin\theta\left[(16\cos^2\theta+1)(16\sin^2\theta+1)\right]'}{(16\cos^2\theta+1)^2(16\sin^2\theta+1)^2}$$

$$=\frac{(16\cos^2\theta+1)(16\sin^2\theta+1)64(-\sin^2\theta+\cos^2\theta) - 64\cos\theta\sin\theta\left[-16(2)\cos\theta\sin\theta(16\sin^2\theta+1) + (16\cos^2\theta+1)(16(2)\sin\theta\cos\theta)\right]}{\text{Bottom}^2}$$

$$=\frac{(16\cos^2\theta+1)(16\sin^2\theta+1)64(-\sin^2\theta+\cos^2\theta) + 2048\cos^2\theta\sin^2\theta(16\sin^2\theta+1) - 2048\cos^2\theta\sin^2\theta(16\cos^2\theta+1)}{\text{bottom}^2}=0$$

$$\overset{+1}{16-16\cos^2\theta}$$

$$\text{Top} = (16\cos^2\theta+1)\big(16(1-\cos^2\theta)+1\big)64 \cdot \big(-(1-\cos^2\theta)+\cos^2\theta\big)$$

$$+\ 2048\cos^2\theta(1-\cos^2\theta)\big(16(1-\cos^2\theta)+1\big)$$

$$-\ 2048\cos^2\theta(1-\cos^2\theta)(16\cos^2\theta+1)$$

$$= \big((16t+1)(17-16t)64(-1+2t)$$

$t=\cos^2\theta$

$$+2048t(1-t)(17-16t)$$

$$-2048t(1-t)(16t+1)=0$$

Time for desmos. $0 \le t \le 1$

$$\Rightarrow t = \tfrac{1}{2},\ 0.072,\ 0.928 = \cos^2\theta$$

$$x = 4\cos\theta = 4\tfrac{1}{\sqrt{2}},\ 4\sqrt{0.072},\ 4\sqrt{0.928}$$

$$x = 2.83,\ 1.07,\ 3.85$$

$$y=\sqrt{16-x^2} = 2.83,\ 3.85,\ 1.07$$

$$\sqrt{16-4/\sqrt{2}}$$

In Region 3 3 possible pts

$(2.83, 2.83)$, $(1.07, 3.85)$

$(3.85, 1.07)$

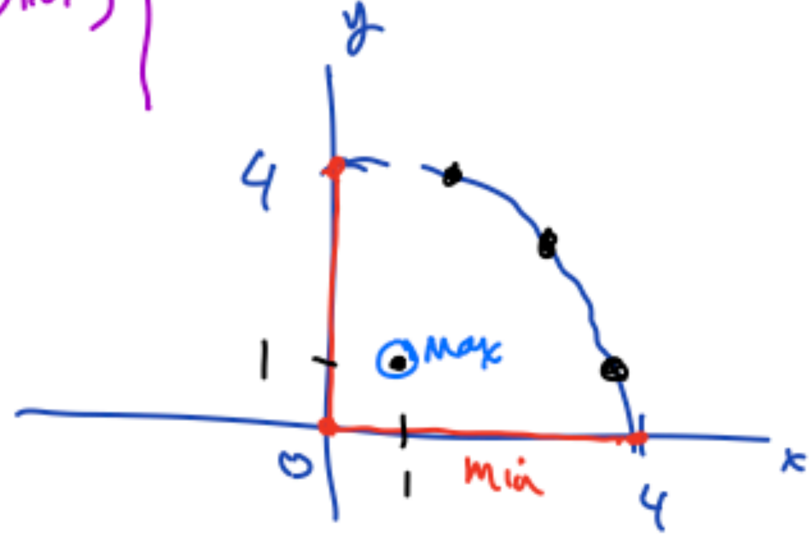
Corner pts: $(0,0)$, $(4,0)$, $(0,4)$

τ values $=0$.

Check for abs max:

$$S(x,y) = \frac{4xy}{(x^2+1)(y^2+1)}$$

(x,y)	$S(x,y)$
$(0,0)$	0 ← abs min
$(4,0)$	0 ← abs min
$(0,4)$	0 ← abs min
$(1,1)$	$\frac{4\cdot1\cdot1}{2\cdot2}=1$ ← abs max
$(x,0)$	0 ← abs min
$(0,y)$	0 ← abs min
$(2.83,2.83)$	$.395$
$(1.07,3.85)$	$.486$
$(3.85,1.07)$	$.486$



The abs min occurs when x or $y=0$, and the abs. min value is 0.

The abs max occurs when $x=y=1$ $(1,1)$, with value 1.

Consider the same function
$$S(x,y) = \frac{4xy}{(x^2+1)(y^2+1)}$$
on the region $A = \{(x,y) : x \le 0, y \ge 0\}$.

Find abs. max & abs min.

unbounded – we are not guaranteed to find abs max & min.

Let's try:

① Interior Cr. Pts — same math as before:

(0,0) ← corner

(-1,1) ← inside, (-1,-1) ← outside

(1,1) ← outside, (1,-1) ← outside

possible cr pts: (0,0), (-1,1).

Boundary pieces (x,0), (0,y) ← $S(0,y)=0$.

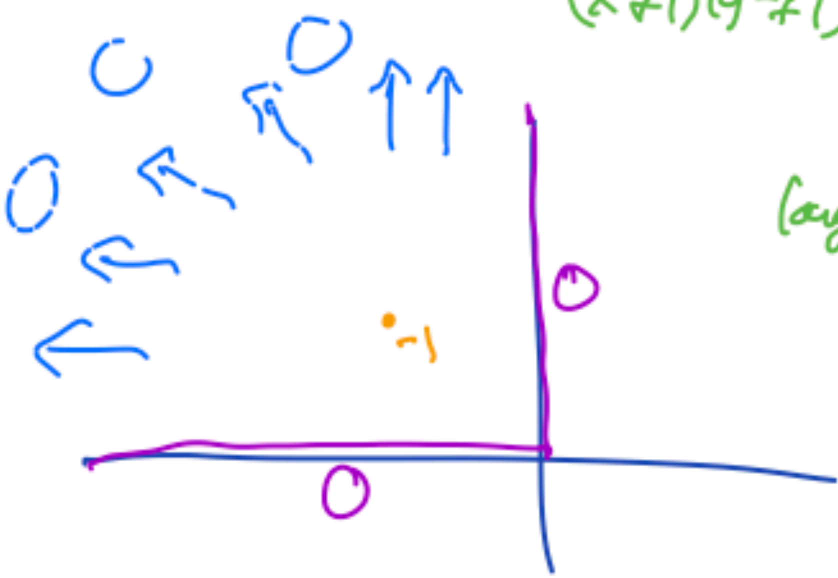
(x,y)	$S(x,y)$
$(0,0)$	0 ← potential max
$(-1,1)$	$\frac{-4}{4} = -1$ ← potential abs. min.
$(x,0)$	0 ←
$(0,y)$	0 ←

$$S(x,y) = \frac{4xy}{(x^2+1)(y^2+1)}$$

In this particular case, we can actually determine what is happening.

$$S(x,y) = \frac{4xy}{(x^2+1)(y^2+1)} = 4\left(\frac{x}{x^2+1}\right)\left(\frac{y}{y^2+1}\right)$$

$\frac{x}{x^2+1} \to 0$, $\frac{y}{y^2+1} \to 0$ when x & y get large in abs value



$S(x,y) \leq 0$. ← must be max value.

If there is another place where $S(x,y) < -1$, at that place we would have to have another cr. pt.

So in this case, even though the region is not compact, we were able to find the abs max

with value 0

& abs min $(-1,1)$ (with value -1).

(Ex) $U(x,y) = xy$, same region

$A = \{(x,y) : x \leq 0,\ y \geq 0\}$.

in this case we get abs max when $x=0$ or $y=0$
$(0,y)$ $(x,0)$.

But there is no abs min.
($U(x,y) = xy \to -\infty$ as $y=1$, $x \to -\infty$.)

Next time: Finding abs max & min of functions on surfaces & curves.

Lagrange multipliers.